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APPLICATION OF WEAK TURBULENCE THEORY TO FPU MODEL

Peter R. Kramer, Joseph A. Biello, and Yuri Lvov

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
301 Amos Eaton Hall
110 8th Street
Troy, NY 12180, USA

Abstract. The foundations of weak turbulence theory is explored through its application to the (alpha) Fermi-Pasta-Ulam (FPU) model, a simple weakly nonlinear dispersive system. A direct application of the standard kinetic equations would miss interesting dynamics of the energy transfer process starting from a large-scale excitation. This failure is traced to an enforcement of the exact resonance condition, whereas mathematically the resonance should be broadened due to the energy transfer happening on large but finite time scales. By allowing for the broadened resonance, a modified three-wave kinetic equation is derived for the FPU model. This kinetic equation produces some correct scaling predictions about the statistical dynamics of the FPU model, but does not model accurately the detailed evolution of the energy spectrum. The reason for the failure seems not to be one of the previously clarified reasons for breakdown in the weak turbulence theory.

1. **Introduction.** Weak turbulence (WT) theory was developed in the 1960s to provide equations which quantitatively describe the transfer of energy among turbulent, weakly nonlinear, dispersive waves in fluids [1, 2, 13, 14, 21, 22, 26]. The basic "kinetic equations" produced by WT theory have been applied to analyze energy transfer include internal and surface waves with small aspect ratios in the atmosphere and ocean [18, 26], semiconductor lasers [17], and plasma turbulence [14]. The domain of validity of the governing equations, however, remains to be fully explored.

Zakharov [26] and Kadomtsev [14] obtained the kinetic equations through arguments (such as random phase approximation) motivated by physical intuition, whereas various authors in the USA and Western Europe [1, 2, 13, 22] derived the same kinetic equations through a multiple scales expansion. While the latter approach is systematic, it has not been rigorously formulated, and several breakdowns of its validity have been noted in recent years. Newell and coworkers [4, 9, 21] have noted that the basic asymptotic expansion parameter is usually not uniformly small over the full range of scales in a weakly turbulent system, so that "intermittent" effects not directly predictable by WT theory will manifest themselves at either sufficiently large or small scales. Cai, Majda, McLaughlin, and Tabak [6, 7, 19] have scrutinized the validity of the kinetic equations of WT theory and their underlying assumptions through their application on a simple one-dimensional model amenable

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to comparison with direct numerical simulation. One key finding of their work is that under certain circumstances, a nonlocal coupling between the large-scale forcing and much smaller "inertial range" scales can arise which invalidates the tacit assumption that the energy transfer in this inertial range of scales is unaffected by the presence of forcing and/or damping.

To further these efforts at understanding the domain of validity of WT theory, we study its application to a simple Hamiltonian model without driving or damping. The Fermi-Pasta-Ulam (FPU) model of a one-dimensional collection of particles coupled by weakly anharmonic springs serves as a natural test example which ostensibly should be amenable to the WT approach, provided the number N of particles in the chain is large enough. Indeed, the FPU model has dispersive waves which are coupled through a weak nonlinearity, and it can be initialized with a disordered wave profile. Moreover, its Hamiltonian nature excludes the problem related to the nonlocal force coupling. We consider here the α version of the FPU model (defined in Section 2) which has a cubic nonlinearity in the Hamiltonian and therefore direct three-wave coupling (which is usually simpler to analyze than four-wave couplings as studied in [6, 7, 19]). Nonetheless, we still find the standard kinetic equations of weak turbulence insufficient to predict the dynamics of the energy spectrum in the FPU model when it is initialized with a large-scale excitation.

As we discuss in Section 4 after some notational preparation in Section 3, there are no nontrivial exact triad (three-wave) resonances. A literal application of the standard WT theory would suggest then that the energy transfer between waves must be governed by four-wave interactions. A scrutiny of the numerical simulations of the α-FPU dynamics [3], however, indicates the presence of at least two distinct time scales on which significant energy transfer occurs. The first phase only involves energy transfer and equilibration among modes of sufficiently small wavenumber. In the final phase, occurring on a much longer time scale, energy is redistributed among all modes. These observations can be reconciled with the resonance properties of the dispersion relations by recognizing that the low wavenumber modes are in *near resonance* because the waves become approximately acoustic at low wavenumber. Consequently, we can interpret the first phase of energy transfer among low wavenumber modes as being mediated by triad interactions and the later phase of relaxation to global energy equipartition as requiring quartet interactions.

We attempt here to adapt weak turbulence theory to describe the energy transfer among nearly resonant triads during the earlier phase of evolution of the energy spectrum in the α-FPU model. Revisiting the multiple scales derivation of the three-wave kinetic equation [22] in Section 5, we identify a step in which a large but finite time is replaced by an infinite time limit. It is this simplification which enforces the exact resonance condition present in the standard kinetic equations. In reality, the resonance should be broadened simply because the three-wave kinetic equation seeks to describe the statistical dynamics of a turbulent system on at a large but finite time. Over a long but finite time T, frequency mismatches of order $o(T^{-1})$ cannot be detected. By properly retaining the effects of this "finite-time resonance broadening" and employing a systematic renormalization group technique, we can compute the range of wavenumbers which will exchange significant energy through triad interactions and the time scale on which this energy exchange should occur. These scaling predictions are well confirmed by the numerical results, and agree with earlier partial predictions [8, 11, 25].

Our calculations also produce a modified kinetic equation (12) with finite-time resonance broadening. Unfortunately, the details of this kinetic equation are not in good agreement with the numerically observed energy spectrum evolution. Moreover, the modified kinetic equation generates negative energies. The problem seems to be the omission of higher order secular terms in the multiple scales expansion which play an important role when the waves are weakly dispersive [16], but does not appear to fall neatly into an established situation where the WT theory is known to fail. In future work, we will attempt to clarify the nature of the breakdown and assess whether a further modified WT approach might succeed.

2. **Fermi-Pasta-Ulam Model.** The Fermi-Pasta-Ulam (FPU) model is a model for a one-dimensional collection of particles with massless, weakly anharmonic (nonlinear) springs connecting them to each other. Letting $\{q_j\}_{j=1}^N$ and $\{p_j\}_{j=1}^N$ denote the position and momentum coordinates of an N-particle chain, we can define the (α version) FPU model Hamiltonian, which we write immediately in nondimensional form:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{j=1}^{N}\left(p_j^2+(q_j-q_{j+1})^2\right)+\frac{\epsilon}{3}\sum_{j=1}^{N}(q_j-q_{j+1})^3 \qquad (1)\\ \frac{\mathrm{d}q_j}{\mathrm{d}t} &= p_j\\ \frac{\mathrm{d}p_j}{\mathrm{d}t} &= (q_{j-1}-2q_j+q_{j+1})\left(1+\epsilon(q_{j-1}-q_{j+1})\right)\end{aligned}$$

The coordinates are normalized so that the total Hamiltonian energy $\mathcal{H} = N$, implying a typical $O(1)$ value of each q_j and p_j. The small parameter $\varepsilon \ll 1$ then measures the weakness of the nonlinearity.

We now make a conformal transformation to action-angle variables to facilitate the application of WT theory. We describe the transformation by the following two steps:

$$\begin{aligned}\begin{pmatrix} Q_k \\ P_k \end{pmatrix} &= \sum_{j=1}^{N}\begin{pmatrix} q_j \\ p_j \end{pmatrix}\exp\left(\frac{2\pi ikj}{N}\right)\\ a_k &= \frac{1}{\sqrt{2\omega_k}}\left(P_k-i\omega_kQ_k\right), \qquad (2)\end{aligned}$$

where $\omega_k \equiv 2\left|\sin\left(\frac{\pi k}{N}\right)\right|$ is the dispersion relation describing the frequency of a discrete Fourier mode k. The virtue of the new (complex-valued) canonical variables a_k is that, in the absence of nonlinearity, they rotate with a single frequency ($a_k \propto \mathrm{e}^{-i\omega_k t}$). To connect with the standard notation of WT theory and keep subsequent expressions as compact as possible, we define the variables $A_k^- \equiv a_k$, and $A_k^+ \equiv a_{-k}^*$. The superscript refers to the branch of the dispersion relation: $A_k^s \propto \mathrm{e}^{is\omega_k t}$.

In these new variables, the Hamiltonian reads

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{N}\sum_{k_1,k_2=-\frac{N}{2}+1}^{\frac{N}{2}}\sum_{s_1,s_2=\pm}\mathcal{H}^{s_1,s_2}_{k_1,k_2}A^{s_1}_{k_1}A^{s_2}_{k_2}\tilde{\delta}^{(N)}_{k_1+k_2,0} \qquad (3)\\
&\quad+\frac{\varepsilon}{N^2}\sum_{k_1,k_2,k_3=-\frac{N}{2}+1}^{\frac{N}{2}}\sum_{s_1,s_2,s_3=\pm}\mathcal{H}^{s_1,s_2,s_3}_{k_1,k_2,k_3}A^{s_1}_{k_1}A^{s_2}_{k_2}A^{s_3}_{k_3}\tilde{\delta}^{(N)}_{k_1+k_2+k_3,0},\\
\mathcal{H}^{s_1,s_2}_{k_1,k_2} &= \frac{1}{2}\sqrt{\omega_{k_1}\omega_{k_2}}\delta_{s_1,-s_2}, \qquad (4)\\
\mathcal{H}^{s_1,s_2,s_3}_{k_1,k_2,k_3} &= \frac{s_1s_2s_3\,\mathrm{sgn}\,(k_1k_2k_3)\sqrt{\omega_{k_1}\omega_{k_2}\omega_{k_3}}}{6\sqrt{2}}. \qquad (5)
\end{aligned}
$$

The symbol $\delta_{i,j}$ denotes the Kronecker delta function and $\tilde{\delta}^{(N)}_{i,j}$ is a periodized version which equals 1 if $i \equiv j \pmod N$ and 0 otherwise. The equations of motion can then be written [22]

$$
\begin{aligned}
\frac{\mathrm{d}A^s_k}{\mathrm{d}t} - is\omega_k A^s_k &= \varepsilon\frac{1}{N}\sum_{s_1,s_2=\pm}\sum_{k_1,k_2=-\frac{N}{2}+1}^{\frac{N}{2}} L^{ss_1s_2}_{kk_1k_2}A^{s_1}_{k_1}A^{s_2}_{k_2}\tilde{\delta}^{(N)}_{k,k_1+k_2}, \qquad (6)\\
L^{ss_1s_2}_{kk_1k_2} &= \frac{is_1s_2\,\mathrm{sgn}\,(kk_1k_2)\sqrt{\omega_k\omega_{k_1}\omega_{k_2}}}{2\sqrt{2}}. \qquad (7)
\end{aligned}
$$

To keep focus, we will restrict the initial data to have statistical symmetry under parity inversion and to have Gaussian statistics. Then it suffices to describe the energy transfer among different scales through the statistical quantities [22]:

$$
\begin{aligned}
n_k \equiv \frac{1}{N}\langle |a_k|^2\rangle &= \frac{1}{N}\langle A^+_k A^-_{-k}\rangle, \qquad (8)\\
E_k \equiv \omega_k n_k + \omega_{-k}n(-k) &= 2\omega_k n_k.
\end{aligned}
$$

The main object of our interest will be E_k, the harmonic energy associated to the wavenumber $\pm k$, or in other words the mean contribution to the quadratic part of the Hamiltonian in (5) from wavenumbers $\pm k$. The mean-square amplitude n_k of the action variables is sometimes referred to as the "occupation number" of the mode k.

3. **Notational Convention.** The weak turbulence theory which we will apply to analyze the FPU model is usually formulated in terms of spatially homogenous systems of infinite extent, so that the Fourier wavenumbers are continuously distributed. This would be appropriate for a particular well-defined $N \to \infty$ limit of the FPU model, in which the physical lattice spacing remains order unity and the system becomes infinitely extended in both directions in physical space. We will always consider N to be finite but large, and introduce notation which makes contact with that usually used in the weak turbulence literature [22].

We define the normalized wavenumber $\xi = k/N$ which takes values in the unit circle S_1 (or equivalently, the space of real numbers $\mathbb{R}$ with all numbers differing by an integer declared equivalent) [5]. To distinguish between Fourier coefficients defined with respect to the integer wavenumbers k and the new wavenumber variable ξ, we write a_k for the former and $a(\xi)$ for the latter. The associated frequency is defined through the continuous dispersion relation $\omega(\xi) \equiv 2\,|\sin \pi\xi|$.

For large but finite N, the correlation functions of the Fourier coefficients are proportional to a discretized delta-function, in the sense that

$$\begin{aligned} \langle a(\xi)a^*(\xi')\rangle &= n(\xi)\tilde{\delta}^{\sharp(N)}(\xi-\xi') \\ \tilde{\delta}^{\sharp(N)}(\xi) &\equiv N\tilde{\delta}^{(N)}_{N\xi,0}. \end{aligned}$$

where $n(\xi)$ has a finite, nontrivial $N \to \infty$ limit.

To further emphasize the connection between the finite but large N and infinitely long systems, we introduce the discretized integral notation:

$$\int_D^{\sharp(M)} f(\xi)\,\mathrm{d}\xi \equiv \frac{1}{M} \sum_{\xi:\xi\in D\cap\mathbb{Z}/M} f(\xi),$$

which should just be thought of as a Riemann integral approximation to the integral defined with respect to a mesh situated at integer multiples of $1/M$.

4. **Modified Kinetic Equation.** The standard weak turbulence theory [22, 26] would predict that three-wave interactions would be incapable of significant energy transfer. The reason is that the only triads of waves which satisfy the exact resonance conditions

$$\xi = \xi_1 + \xi_2, \qquad \omega(\xi) = \pm\omega(\xi_1) \pm \omega(\xi_2)$$

are those involving the zero mode ($\xi = 0$ or $\xi_1 = 0$ or $\xi_2 = 0$), which is uncoupled to the other degrees of freedom due to the translational symmetry of the system ($L_{\xi\xi_1\xi_2} = 0$ if $\xi\xi_1\xi_2 = 0$). The absence of three-wave resonances is here not due to the finite size of the system [10] but simply due to the concavity of the dispersion relation [24].

The reason the standard kinetic equations of weak turbulence theory fail to describe any energy transfer through three-wave interactions is that these interactions are effective only over a thin band of wavenumbers ξ with width vanishing with the nonlinearity parameter ε. The derivation of the standard weak turbulence kinetic equations assumes that all quantities, including the wavenumbers ξ of interest and the energy density $E(\xi)$ in each mode, are order unity in magnitude (asymptotically independent of ε). The fact that the wavenumbers of interest during the three-wave dynamics are small, and the energy density residing in these wavenumbers large, requires a reworking of the asymptotic derivation of the kinetic equation.

To this end, a simple renormalization group approach is useful [12, 20]. We rescale the wavenumber and amplitude of the energy spectrum by some gauge functions $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ depending on ε:

$$\bar{\xi} = \xi/\beta(\varepsilon), \qquad \bar{t} = t/\gamma(\varepsilon). \tag{9}$$

The rescaling factors β and γ are chosen so that $\bar{\xi}$ and $\bar{t}$ are order unity for the three-wave dynamics of interest. We shall leave them unspecified for now, other than insisting that $\beta \downarrow 0$ and $\gamma \to \infty$ as $\varepsilon \to 0$, and show how their scaling with ε can be determined by self-consistency. More specifically, we define a rescaled energy spectrum $E^{(\varepsilon)}(\bar{\xi},\bar{t})$, obtained by by applying the transformation (9) to the original energy spectrum:

$$E(\xi,t) = \beta(\varepsilon)^{-1}E^{(\varepsilon)}(\xi/\beta(\varepsilon), t/\gamma(\varepsilon)) \tag{10}$$

The rescaling of the amplitude results from the normalization condition that the total quadratic energy is N, which implies that if $E(\xi,t)$ is concentrated over a thin band of wavenumbers $|\xi| \lesssim \beta(\varepsilon)$, then the average amplitude of the energy spectrum over these modes must be $\mathrm{ord}\,(\beta(\varepsilon)^{-1})$. The rescaled energy spectrum $E^{(\varepsilon)}(\bar{\xi},\bar{t})$ is supposed to be a truly order unity function, with order unity amplitude and order unity variations with respect to $\bar{\xi}$ and $\bar{t}$ during the three-wave dynamical phase for all sufficiently small ε. Mathematically, we demand that the *renormalized energy spectrum*

$$\bar{E}(\bar{\xi},\bar{t}) = \lim_{\varepsilon\to 0} E^{(\varepsilon)}(\bar{\xi},\bar{t})$$

be a nontrivial function with order unity amplitude and derivatives. In renormalization group terminology, we want $\bar{E}(\bar{\xi},\bar{t})$ to be a nontrivial fixed point under the transformation induced by (10). The gauge functions $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ are determined by this property. Indeed, as we shall show in Section 5,

$$\beta = \varepsilon^{1/2}, \qquad \gamma = \varepsilon^{-3/2}. \tag{11}$$

and the renormalized energy spectrum satisifes the following modified kinetic equation:

$$\frac{\partial \bar{E}(\bar{\xi},\bar{t})}{\partial \bar{t}} = \frac{1}{\pi}\int_{\mathbb{R}}^{\sharp(\varepsilon^{1/2}N)}\int_{\mathbb{R}}^{\sharp(\varepsilon^{1/2}N)} \frac{\sin(\pi^3\bar{\xi}\bar{\xi}_1\bar{\xi}_2\bar{t})}{\bar{\xi}_1\bar{\xi}_2}\tilde{\delta}^{\sharp(\varepsilon^{1/2}N)}(\bar{\xi}-\bar{\xi}_1-\bar{\xi}_2) \tag{12}$$

$$\times\left[\bar{\xi}\bar{E}(\bar{\xi}_1,\bar{t})\bar{E}(\bar{\xi}_2,\bar{t}) - \bar{\xi}_1\bar{E}(\bar{\xi},\bar{t})\bar{E}(\bar{\xi}_2,\bar{t}) - \bar{\xi}_2\bar{E}(\bar{\xi},\bar{t})\bar{E}(\bar{\xi}_1,\bar{t})\right]\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\bar{\xi}_2 \tag{13}$$

We comment now on these predictions.

4.1. **Scaling Predictions.** The self-consistency calculation based on the weak turbulence expansion predicts that the band of wavenumbers participating in the earliest (three-mode) phase of energy transfer has thickness $\Delta k \sim \beta = \varepsilon^{1/2}$ and associated dynamical time scale $t \sim \gamma = \varepsilon^{-3/2}$, provided the initial data is distributed over a band of low wavenumbers which is comparably $(\varepsilon^{1/2})$ thick. In [3], these scaling predictions are confirmed through direct numerical simulations. Analytical explanations for the $\varepsilon^{1/2}$ width for the band of wavenumbers which equilibrate first have been previously offered on the basis of resonance overlap calculations [25] and related considerations [8, 11]. One can also construct an argument for the $\varepsilon^{-3/2}$ dynamical time scale from cruder arguments. Our intent, however, was to provide a more systematic argument, since resonance overlap arguments can sometimes give misleading results [15]. As we discuss below, however, we cannot yet claim to have fulfilled this goal.

4.2. **Properties of Modified Kinetic Equation.** The modified kinetic equation (12) has the following good properties:

- The modified three-wave kinetic equation (12) conserves the harmonic energy over the three-wave time scale:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_1}^{\sharp(N)} E(\xi,t)\,\mathrm{d}\xi = 0,$$

  More precisely, direct calculations on the modified kinetic equation (12) show that the three-wave interactions conserve the harmonic energy locally (triad by triad) for the band of wavenumbers $\xi \lesssim \varepsilon^{1/2}$, while the energy in wavenumbers $\xi \gg \varepsilon^{1/2}$ is predicted not to change significantly on the time scale $t \lesssim \varepsilon^{-3/2}$.

- The modified kinetic equation accounts for a finite time resonance broadening in that the usual Dirac delta function $\delta(\Delta\omega)$ involving the frequency difference (16) is instead replaced in (12) by a renormalized form of the function $\frac{\sin(\Delta\omega t)}{\Delta\omega}$. While the latter does indeed converge to the Dirac delta function in the $t \to \infty$ limit, its distinction from this strict limit is important for predicting nontrivial dynamics for large but finite t. Some previous work [10, 23] offered *ad hoc* modifications to the weak turbulence theory to take into account such broadening of the resonances, but we have endeavored to derive the quantitative form of the finite-time resonance broadening through systematic means. In particular, we naturally link the width of the resonance broadening to the inverse of the dynamical time scale of observational interest, $t\varepsilon^{-3/2}$.

4.3. **Deficiency in Modified Kinetic Equation.** The modified kinetic equation, however, has a serious deficiency in that it does not preserve nonnegativity of the energy spectrum, because the broadened resonance kernel $\sin(\pi^3\bar{\xi}\bar{\xi}_1\bar{\xi}_2\bar{t})/(\bar{\xi}\bar{\xi}_1\bar{\xi}_2)$ is not sign-definite. This indicates that, unlike standard weak turbulence theory, the removal of the leading order secular term (along with frequency renormalization) does not simultaneously remove all higher order secularities active at the first nonlinear time scale, $t \sim \gamma = \varepsilon^{-3/2}$, in the α-FPU model. A similar issue arose in the problem of acoustic turbulence [16].

Newell and coworkers [4, 9, 21] have recently pointed out that the assumptions underlying the WT theory will often fail at either large or small scales, because the nonlinear time scale t_{NL} associated with such scales may fail to be long compared with the corresponding linear time scale t_{L}. Because the development of partial equipartition is concentrated on small wavenumbers $k \lesssim \beta \sim \varepsilon^{1/2}$, one might think that the deficiencies noted for the kinetic equation derived above arise from this kind of breakdown in the asymptotic expansion. We now show, however, that the simplest such explanation is insufficient.

The linear time scale t_{L} associated with a wavenumber ξ is just the inverse frequency $(\omega(\xi))^{-1}$, while the associated nonlinear time scale $t_{\mathrm{NL}} \sim \frac{\partial E(\xi,t)/\partial t}{E(\xi,t)}$. From studying either the primitive second order WT expansion (14) or the renormalized kinetic equation, we see that for the band of active wavenumbers $\xi \lesssim \varepsilon^{1/2}$, the nonlinear time scale depends on ξ and ε as $t_{\mathrm{NL}} \sim (\varepsilon\omega(\xi))^{-1}$. Therefore, for $\xi \lesssim \varepsilon^{1/2}$, the ratio $t_{\mathrm{NL}}/t_{\mathrm{L}} \sim \varepsilon^{-1}$ is uniformly large, so the defects in the kinetic equation cannot be attributed to this criterion elaborated in [4, 9, 21].

Another situation in which weak turbulence theory will fail is in finite systems where the wavenumber spacing $1/N$ is not small compared with the width of the frequency resonance, which is here proportional to $\beta \sim \varepsilon^{1/2}$ [22]. Therefore, WT theory cannot be expected to apply to the α-FPU model when $N\varepsilon^{1/2} \lesssim 1$, but this can be avoided, at least in principle, by choosing the size of the system N to be large enough.

At this time, we are endeavoring to specify concretely the source of the deficiency of the modified kinetic equation.

5. **Derivation of Modified Kinetic Equation.** Our starting point will be the following equation, which results from a multiple-scales expansion for the moments

of the mode amplitudes a [22]:

$$\begin{aligned}
\frac{\partial n(\xi,t)}{\partial t} &= -4\varepsilon^2\Re\sum_{s_1,s_2=\pm}\int_{S_1}^{\sharp(N)}\int_{S_1}^{\sharp(N)}\mathcal{K}^{(t)}(\omega(\xi)-s_1\omega(\xi_1)-s_2\omega(\xi_2)) \\
&\quad\times L^{+s_1s_2}_{\xi\xi_1\xi_2}\Big[L^{+,s_1,s_2}_{\xi,\xi_1,\xi_2}n(\xi_1,t)n(\xi_2,t)+L^{-s_1,s_2,-}_{-\xi_1,\xi_2,-\xi}n(\xi_2,t)n(\xi,t) \\
&\qquad +L^{-s_2,-,s_1}_{-\xi_2,-\xi,\xi_1}n(\xi,t)n(\xi_1,t)\Big] \\
&\quad\times\tilde{\delta}^{\sharp(N)}(\xi-\xi_1-\xi_2)\mathrm{e}^{i(-\omega(\xi)+s_1\omega(\xi_1)+s_2\omega(\xi_2))t}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&\quad+\ldots
\end{aligned}\tag{14}$$

Here we have used the symbol $\Re$ to denote the real part of the ensuing expression, and defined the function

$$\mathcal{K}^{(t)}(\varpi)=\frac{\mathrm{e}^{i\varpi t}-1}{i\varpi}.$$

The ellipses denote terms which are omitted because they are formally of higher order in ε. We will revisit the effects of these neglected terms in future work, because terms which are insignificant in standard weak turbulence theory must be carefully examined to see if they remain insignificant in our modified derivation.

We convert Eq. (14) into an equation for the energy spectrum through the relation $E(\xi)=2\omega(\xi)n(\xi)$ and the explicit formula (7) for the coupling coefficient. Then upon substituting the rescaling (10) into (14), we obtain

$$\begin{aligned}
\frac{\partial E^{(\varepsilon)}(\bar{\xi},\bar{t})}{\partial\bar{t}} &= -4\varepsilon^2\gamma\Re\sum_{s_1,s_2=\pm}\int_{\beta^{-1}S_1}^{\sharp(\beta N)}\int_{\beta^{-1}S_1}^{\sharp(\beta N)}\gamma\mathcal{K}^{(\bar{t})}(\gamma\Delta\omega) \\
&\quad\times\tilde{L}^{+s_1s_2}_{\beta\bar{\xi},\beta\bar{\xi}_1,\beta\bar{\xi}_2}\Big[\tilde{L}^{+,s_1,s_2}_{\beta\bar{\xi},\beta\bar{\xi}_1,\beta\bar{\xi}_2}E^{(\varepsilon)}(\bar{\xi}_1,\bar{t})E^{(\varepsilon)}(\bar{\xi}_2,\bar{t}) \\
&\qquad+\tilde{L}^{-s_1,s_2,-}_{-\beta\bar{\xi}_1,\beta\bar{\xi}_2,-\beta\bar{\xi}}E^{(\varepsilon)}(\bar{\xi}_2,\bar{t})E^{(\varepsilon)}(\bar{\xi},\bar{t}) \\
&\qquad+\tilde{L}^{-s_2,-,s_1}_{-\bar{\xi}_2,-\bar{\xi},\bar{\xi}_1}E^{(\varepsilon)}(\bar{\xi},\bar{t})E^{(\varepsilon)}(\bar{\xi}_1,\bar{t})\Big] \\
&\quad\times\tilde{\delta}^{\sharp(\beta N)}(\bar{\xi}-\bar{\xi}_1-\bar{\xi}_2)\mathrm{e}^{-i\gamma\Delta\omega\bar{t}}\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\bar{\xi}_2 \\
&\quad+\ldots
\end{aligned}\tag{15}$$

where

$$\Delta\omega\equiv\omega(\beta\bar{\xi})-s_1\omega(\beta\bar{\xi}_1)-s_2\omega(\beta\bar{\xi}_2)\tag{16}$$

and

$$\tilde{L}^{ss_1s_2}_{\xi\xi_1\xi_2}=\sqrt{\frac{\omega(\xi)}{\omega(\xi_1)\omega(\xi_2)}}L^{ss_1s_2}_{\xi\xi_1\xi_2}=\frac{is_1s_2\,\mathrm{sgn}\,(kk_1k_2)\omega(\xi)}{2\sqrt{2}}.$$

We now seek particular relations between γ, β, and ε so that the right hand side of (15) is consistent with the assumed order unity magnitude and scale of variation of the left hand side in the limit $\varepsilon\to 0$. To this end, we recall that $\beta(\varepsilon)\downarrow 0$, $\gamma(\varepsilon)\to\infty$, and observe that

$$\begin{aligned}
\omega(\xi) &= \omega(\beta\bar{\xi}) \sim 2\pi\beta|\bar{\xi}| + O(\beta^3), \\
\Delta\omega &= 2\pi\beta(|\bar{\xi}| - s_1|\bar{\xi}_1| - s_2|\bar{\xi}_2|) - \frac{\pi^3\beta^3}{3}(|\bar{\xi}|^3 - s_1|\bar{\xi}_1|^3 - s_2|\bar{\xi}_2|^3) + O(\beta^5) \\
&= \pi\beta(|\bar{\xi}| - s_1|\bar{\xi}_1| - s_2|\bar{\xi}_2|) \\
&\qquad \times \left[2 - \frac{\pi^2\beta^2}{3}\left(|\bar{\xi}|^2 + |\bar{\xi}|(s_1|\bar{\xi}_1| + s_2|\bar{\xi}_2|) + \bar{\xi}_1^2 + \bar{\xi}_2^2 - s_1 s_2|\bar{\xi}_1||\bar{\xi}_2|\right)\right] \\
&\quad -\pi^3\beta^3 s_1 s_2|\bar{\xi}||\bar{\xi}_1||\bar{\xi}_2| + O(\beta^5).
\end{aligned}$$

Noting that $\bar{\xi} = \bar{\xi}_1 + \bar{\xi}_2$ from the Kronecker delta function in (15), the appropriate asymptotics for $\Delta\omega$ can be decided based on the s_1, s_2, and the signs of $\bar{\xi}$, $\bar{\xi}_1$, and $\bar{\xi}_2$. If the signs align so that $|\bar{\xi}_1 + \bar{\xi}_2| \equiv s_1|\bar{\xi}_1| + s_2|\bar{\xi}_2|$, then $\Delta\omega \sim \operatorname{ord}(\beta^3)$; otherwise $\Delta\omega \sim \operatorname{ord}(\beta)$. Moreover,

$$\begin{aligned}
&e^{-i\gamma\Delta\omega\bar{t}}\mathcal{K}^{(\bar{t})}(\gamma\Delta\omega) \\
&= \begin{cases} \gamma^{-1}\beta^{-3}\left[\frac{e^{-i\gamma\Delta\omega\bar{t}}-1}{i\pi^3 s_1 s_2|\bar{\xi}||\bar{\xi}_1||\bar{\xi}_2|} + O(\beta^2)\right] & \text{if } |\bar{\xi}_1 + \bar{\xi}_2| = s_1|\bar{\xi}_1| + s_2|\bar{\xi}_2|, \\ \gamma^{-1}\beta^{-1}\left[\frac{1-e^{-i\gamma\Delta\omega\bar{t}}}{2\pi i(|\bar{\xi}|-s_1|\bar{\xi}_1|-s_2|\bar{\xi}_2|)} + O(\beta^2)\right] & \text{otherwise .}\end{cases}
\end{aligned}$$

Therefore, the dominant contribution will come from the wavenumbers in the domain such that $|\bar{\xi}| = |\bar{\xi}_1 + \bar{\xi}_2| = s_1|\bar{\xi}_1| + s_2|\bar{\xi}_2|$. We will choose the scaling functions so that $\Delta\omega \lesssim \gamma^{-1}$ for order unity values of $\bar{\xi}$, $\bar{\xi}_1$, and $\bar{\xi}_2$ in this dominant region, so we can simplify further:

$$\begin{aligned}
e^{-i\gamma\Delta\omega\bar{t}}\mathcal{K}^{(\bar{t})}(\gamma\Delta\omega) &= \gamma^{-1}\beta^{-3}\left[\frac{e^{i\pi^3\gamma\beta^3 s_1 s_2|\bar{\xi}||\bar{\xi}_1||\bar{\xi}_2|\bar{t}} - 1}{i\pi^3 s_1 s_2|\bar{\xi}||\bar{\xi}_1||\bar{\xi}_2|} + O(\beta^2 + \beta^5\gamma)\right] \\
&\qquad \text{when } |\bar{\xi}_1 + \bar{\xi}_2| = s_1|\bar{\xi}_1| + s_2|\bar{\xi}_2|.
\end{aligned}$$

Discarding subdominant contributions based on the above considerations, we achieve after some analytical manipulations:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\partial E^{(\varepsilon)}(\bar{\xi},\bar{t})}{\partial\bar{t}} &= \lim_{\varepsilon\to 0}\frac{\varepsilon^2\gamma}{\pi\beta(\varepsilon)}\int_{\mathbb{R}}^{\sharp(\beta N)}\int_{\mathbb{R}}^{\sharp(\beta N)}\frac{\sin(\beta^3\gamma\pi^3\bar{\xi}\bar{\xi}_1\bar{\xi}_2\bar{t})}{\bar{\xi}_1\bar{\xi}_2} \\
&\quad \times\left[\bar{\xi}\bar{E}(\bar{\xi}_1,\bar{t})\bar{E}(\bar{\xi}_2,\bar{t}) - \bar{\xi}_1\bar{E}(\bar{\xi},\bar{t})\bar{E}(\bar{\xi}_2,\bar{t}) - \bar{\xi}_2\bar{E}(\bar{\xi},\bar{t})\bar{E}(\bar{\xi}_1,\bar{t})\right] \\
&\quad \times\tilde{\delta}^{\sharp(\beta N)}(\bar{\xi} - \bar{\xi}_1 - \bar{\xi}_2)\,d\bar{\xi}_1\,d\bar{\xi}_2
\end{aligned}$$

The equating of the limit of the integral on the right hand side by the integral of the limiting integrand can be justified through uniform bounds on finite intervals and the relative negligibility of the contribution from large $\bar{\xi}_1$ and $\bar{\xi}_2$. The latter fact can be checked by direct analysis or more intuitively understood from the observation that the wavenumbers with $\operatorname{ord}(1)$ magnitude ξ (and therefore large values of $\bar{\xi} = \xi\beta^{-1}$) exhibited no activity on the three-mode interaction time scale.

The gauge functions $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ were introduced so that $E^{(\varepsilon)}(\bar{\xi},\bar{t})$ would have a nontrivial dynamics with order unity magnitude in the $\varepsilon \to 0$ limit. We see then that a nontrivial, finite dynamics results precisely if $\beta^3\gamma$ and $\varepsilon^2\gamma\beta^{-1}$ have finite, nonzero limits. We thereby arrive at the unique (up to irrelevant multiplicative constants) choice of gauge functions (11) and modified kinetic equation (12) for the renormalized energy spectrum.

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E-mail address: `kramep@rpi.edu`