

Rensselaer Polytechnic Institute
Department of Mathematical Sciences
Preliminary Examination
January, 2005

Directions. Please work 10 of the 12 problems, making sure to indicate on the front of the exam book which problems you want read. You have 4 hours to complete the exam. The use of books, notes, calculators, etc. is not permitted during the exam. Answers without complete justification may not receive full credit.

Good luck.

1. A vector field in \mathbb{R}^3 is represented by the vector-valued function

$$\mathbf{F} = \langle -y, x, 9 \rangle, \quad (x, y, z) \in \mathbb{R}^3.$$

Find the flux of the vector field across that portion of a hemispherical surface modeled by

$$z = \sqrt{9 - x^2 - y^2}, \quad 0 \leq x^2 + y^2 \leq 4$$

in the “outward” direction. (The term “outward” means in a direction normal to the surface of the (solid) sphere, $x^2 + y^2 + z^2 \leq 9$, and directed away from its interior.) Note: The disc in the x, y -plane has radius 2 and not 3.

2. Let L denote the linear operator on \mathbb{R}^2 that reflects \mathbb{R}^2 through the line modeled by $y = 2x$, $(x, y) \in \mathbb{R}^2$. Find the matrix of L with respect to the standard basis for \mathbb{R}^2 .

3. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

The distinct eigenvalues of A are 3 and -1 .

- A. Find a non-singular matrix, V , and a diagonal matrix D , such that $A = VDV^{-1}$.
- B. What are the eigenvalues of A^4 ?
- C. What are the dimensions of the eigenspaces corresponding to the eigenvalues found in part B?

4. A. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

The distinct eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. Let I denote the three-by-three identity matrix over \mathbb{R} . Display explicitly the three-by-three matrix

$$A^3 - 5A^2 + 3A + 10I.$$

- B. In \mathbb{R}^2 , two vectors \mathbf{v}_1 and \mathbf{v}_2 are given by $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, -3)$. Let L denote the linear operator on \mathbb{R}^2 such that $L(\mathbf{v}_1) = -\mathbf{v}_1$ and $L(\mathbf{v}_2) = 2\mathbf{v}_2$.

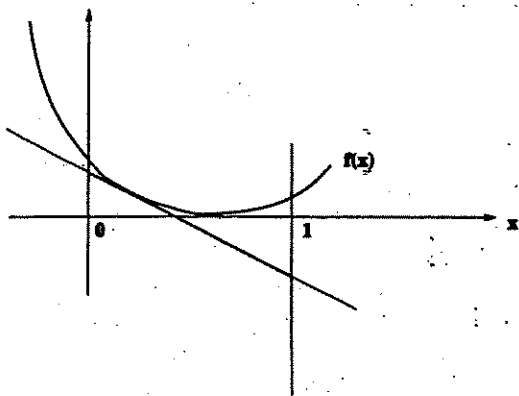
Find the matrix of L with respect to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

- 5. Consider a rectangular box with fixed volume, V , ($V > 0$). Show how to choose the three dimensions of the box, l, w and h , in order to minimize the surface area of the box. (In showing that your values of l, w, h do indeed result in an absolute minimum, you may need to be specific about the domain of definition of the functions you work with.)
- 6. The point $(x_0, y_0, z_0) = (4, 4, 1)$ lies on the surface modeled by

$$y = x(2z - 1), \quad (x, y, z) \in \mathbb{R}^3.$$

Find a model for the tangent plane to the surface at the point (x_0, y_0, z_0) . Your model should be in the form $ax + by + cz + d = 0$, where a, b, c, d denote real numbers.

7. Let f denote a twice continuously differentiable real-valued function defined on $[0, 1]$ such that $f''(x) > 0, \forall x \in [0, 1]$. For each $\xi \in [0, 1]$ let $l_\xi(x), x \in \mathbb{R}$, denote the tangent line to f at $(\xi, f(\xi))$. Find all values of $\xi \in [0, 1]$ such that the area between the graph of f and the tangent line, $l_\xi(x), x \in [0, 1]$, attains its minimum value.



8. Let f denote an infinitely differentiable real-valued function defined on \mathbb{R} . Suppose that for some integer, n

$$0 = f(1) = f(0) = f'(0) = f^{(2)}(0) = \dots = f^{(n)}(0).$$

Argue that there exists $\xi \in (0, 1)$ such that $f^{(n+1)}(\xi) = 0$.

9. Define the two-by-three matrix, A , by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

and a subspace of W of \mathbb{R}^3 by

$$W = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}.$$

Find an explicit form for the three-by-three matrix, P , such that for every $\mathbf{v} \in \mathbb{R}^3$, the vector $P\mathbf{v}$ is the best approximation to \mathbf{v} from W . (It is understood that the metric is defined by the standard Euclidean distance.)

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = e^{-x^2}$, $x \in \mathbb{R}$.
- A. Does there exist a real-valued function, g , defined on \mathbb{R} such that $g'(x) = f(x)$, $\forall x \in \mathbb{R}$? If so, display such a function. If not, provide a convincing argument.
 - B. Estimate the numerical value of $\int_0^1 e^{-x^2} dx$ with an error of less than $\frac{1}{400}$.
11. Let $f : [1, \infty) \rightarrow \mathbb{R}$ and assume that f is continuous, positive and non-increasing.
- A. Assume that $\int_1^\infty f(x) dx$ diverges. Argue that $\sum_{k=1}^\infty f(k)$ diverges. You may want to include a sketch of the graph of such a function.
 - B. Assume that $\int_1^\infty f(x) dx$ converges. Argue that $\sum_{k=1}^\infty f(k)$ converges. You may want to include a sketch of the graph of such a function.
 - C. Show by example that neither A nor B is true without the hypothesis that f is non-increasing.
12. A cylindrical tank is 16 feet high and has a radius of 5 feet. The height of the water in the tank is 10 feet. The bottom of the tank is moveable; it acts as a plunger. It pushes the column of water from the bottom of the tank toward the top. Compute the work done by the plunger (the bottom of the tank) in the process of pushing all of the water over the top of the tank. Water weighs 62.5 lbs/ft³.