Preliminary Examination

Department of Mathematical Sciences
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Notes:

- No books, notes or calculators are allowed.
- Please do any 10 problems. All problems are weighted equally.
- On the front page of the answer book(s), identify the 10 problems you wish graded. Only the 10 problems that you indicate will be considered.
- You have 4 hours to complete the exam.
- Show all work and justify your answers.
- In some cases, answers to an earlier part of a problem may provide helpful hints on how to solve later parts of a problem.

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1. Consider the function

$$f(x,y) = 2x^2 - 6y + 2y^3 + 1$$

   (a) Find and classify all of the critical points of $f(x,y)$.

   (b) Find the absolute maximum and minimum values of $f(x,y)$ on the set $x^2 + 4y^2 \leq 16$

2. The point, $P = (0,0)$, lies on the circle, C, in $\mathbb{R}^2$ modeled by

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1.\}$$

   As C is rolled along the positive x-axis, the point, P, moves on an arc of a cycloid, modeled parametrically by

$$P(t) = (x(t), y(t)),\ 0 \leq t \leq 2\pi.$$

   Here, the parameter t measures displacement of the x coordinate of the center of the circle as it rolls. Express explicitly as a real number, the length of the arc of the cycloid.

3. Evaluate the surface integral

$$\Omega = \iint_S \mathbf{F} \cdot \mathbf{dS}$$

   where $\mathbf{F} = \langle x + e^{yz}\,,\, x^3 \sin(z^5)\,,\, z + 3\cos(y + x^7)\rangle$ and S is the portion of the surface of the sphere $x^2 + y^2 + z^2 = 9$ with $x \geq 0$. Also, $\mathbf{dS}$ is oriented using the outward normal to S.

4. Calculate $\mathbf{A}^{149}$ if $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Your answer should be in the form of a 2×2 matrix.

5. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$.

   (a) Show that V is a subspace of $\mathbb{R}^3$.

   (b) Find a basis for V.

   (c) What is the dimension of V?

6. Let f, g and h denote real-valued functions defined and continuous on the compact real interval, [a,b], and differentiable on the open interval, (a, b).

   (a) Argue that there exists $c \in (a, b)$ such that

$$\det \begin{pmatrix} f(a) & f(b) \\ g(a) & g(b) \end{pmatrix} = (b-a) \det \begin{pmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{pmatrix}.$$

   The symbol, det, denotes determinant.

   (b) Argue that there exists $d \in (a, b)$ such that

$$\det \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(d) & g'(d) & h'(d) \end{pmatrix} = 0.$$

7. Let W denote the real vector space consisting of all continuous real-valued functions defined on the real interval, $[0, 1]$. Addition and scalar multiplication are defined by:

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \ \forall x \in [0,1], \ \forall f, g \in W, \\ (\lambda f)(x) &= \lambda f(x), \ \forall x \in [0,1], \ \forall f \in W, \ \forall \lambda \in \mathbb{R}. \end{aligned}$$

   An inner product and norm are defined on W by the two equations:

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 f(x) g(x)\, dx, \ \forall f, g \in W, \\ \|f\|^2 &= \langle f, f \rangle, \ \forall f \in W. \end{aligned}$$

   Let $f_1, f_2, h \in V$ be defined by

$$f_1(x) = x, \qquad f_2(x) = \cos(2\pi x), \qquad h(x) = 1, \qquad 0 \le x \le 1.$$

   Find an explicit form for the orthogonal projection of h on the span of f_1, and f_2.

8. (a) Find the value of α for which the integral

$$I = \int_1^\infty \left(\frac{x}{x^2+1} - \frac{\alpha}{2x+3} \right) dx$$

converges, and evaluate the integral in that case.

(b) Evaluate the limit

$$L = \lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x$$

9. A tank initially contains 50 gal of pure water. Brine containing γ pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min, and the well-stirred mixture is drained from the tank at the same rate.

(a) Find the amount of salt $Q(t)$ in the tank at any time t.

(b) Find γ if there are 5 lb of salt in the tank after one hour.

10. Determine whether each of the following series converge or diverge.

$$\text{(a)} \quad \sum_{k=1}^\infty \frac{\sqrt{k+1}-\sqrt{k}}{k}, \qquad \text{(b)} \quad \sum_{k=2}^\infty \frac{(\ln k)^3}{(\ln 3)^k}$$

11. (a) Let

$$A = \begin{pmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{pmatrix}.$$

Find an explicit form of a lower triangular matrix, L, and an upper triangular matrix, U, that has $1's$ on the diagonal, such that $A = LU$.

(b) Let

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Argue that the matrix, B, is non-singular and that it does not have an LU decomposition.

12. Let $F(x, y, z) = z - 4xy + 1$ and $G(x, y, z) = \exp(yz) - \cos(x + 2z)$, and let P be the point $(x, y, z) = (2, 0, -1)$.

(a) Find the rate of change of F at P in the direction $\langle 1, 2, -2 \rangle$.

(b) Find a vector tangent to the curve of intersection of the surfaces $F = 0$ and $G = 0$ at P.