Preliminary Examination
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Spring 2003

Instructions:

- No calculators, PDAs, computers, books, or cellular phones are allowed. Do not collaborate in any way.
- Please do any 10 problems. All problems are weighted equally; on multi-part questions, all parts are weighted equally.
- On the front page of your answer book(s), identify the 10 problems you wish graded. Only the 10 problems that you indicate will be considered.
- In order to receive credit, your answers must be clear, legible, and coherent. Show all your work, and justify your answers.

Problems:

1. Suppose $x+y+z=-1$ is a plane and $\mathbf{r}(t)=(t^2,t,-t)$ defines a curve in three dimensions.

   a) Show that the curve $\mathbf{r}$ never intersects the plane.

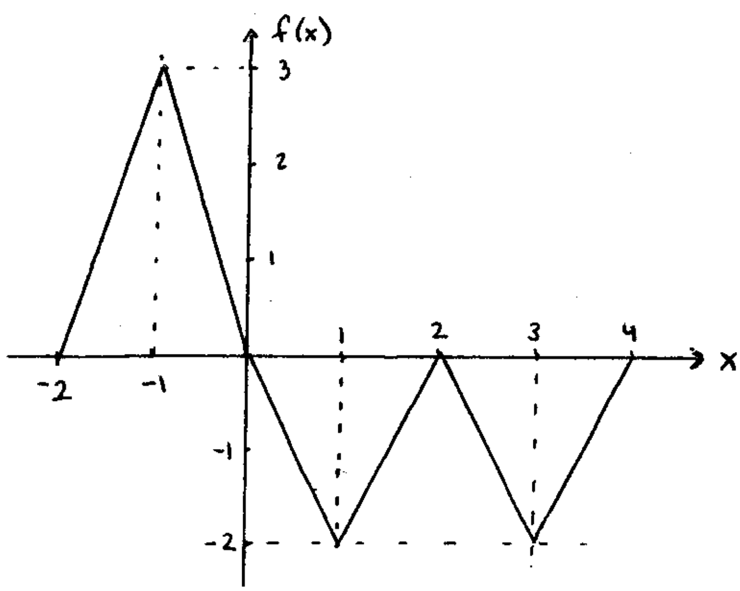
   b) Find the point on the curve that is closest to the plane.

2. Let $f(x)$ be the piecewise linear function pictured below; let $F(t) = \int_{-2}^{t} f(x)\,dx$.

   a) Calculate $F(4)$.

   b) Calculate $F'(2)$.

   c) Draw a rough sketch of $F(t)$ over $[-2,4]$.



3. a) Find all values of x for which the power series $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k}$ converges.

b) Consider the power series

$$\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{2^3} + \frac{x^4}{2^3} + \frac{x^5}{2^5} + \frac{x^6}{2^5} + \cdots$$

Show that this series converges for $|x| < 2$ and diverges otherwise.

4. The temperature at the point (x, y) on the floor of a certain room is given by

$$T(x,y) = 80 - \frac{x^2}{16} - \frac{y^2}{4},$$

where x and y are given in feet. A cat at position $(x, y) = (8, 2)$ is trotting in the direction specified by $\nabla T(8, 2)$, the gradient of T at the point (8,2).

a) If the cat travels at a speed of 3 feet per second, how fast is its local temperature changing at the point $(8, 2)$?

b) If the cat continues moving along a line in the direction $\nabla T(8, 2)$, at what point does it find the warmest spot?

5. Evaluate $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{0}^{\sqrt{x^2+y^2}} (z + x)\ dzdydx$. (Hint: change variables.)

6. Find $\partial z/\partial x$ for the set determined by the system of equations

$$\begin{aligned} \cos x + \log(yz) &= 1 \\ e^{xy} + \sin(xz) &= 1. \end{aligned}$$

Do these equations determine z as a function of x in a neighborhood of $(x, y, z) = (0, 1, 1)$?

7. The number of mosquitos that hatch on a certain island is proportional to the current mosquito population. In the absence of any predation, the population doubles each week. If there are initially 200,000 mosquitos on the island and predators (birds, etc.) eat 20,000 mosquitos per day, how many mosquitos are on the island at time t?

8. Determine the matrix M such $AMB = C$, when

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

9. Let

$$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

(a) Evaluate A^2 and A^3.
(b) Show that $A^2 = A^{-1}$.
(c) Evaluate A^n for any positive integer n.

10. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(a) Show that A has fewer than three linearly independent eigenvectors.
(b) Does $B = A + A^T$ have three linearly independent eigenvectors?

11. Let A be a real symmetric 3×3 matrix that induces the quadratic form $q(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle$, where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ and

$$q(\mathbf{x}) = x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2.$$

a) Find a representation for A.
b) Obtain an orthogonal matrix Q such that the substitution $\mathbf{x} = Q\mathbf{x}'$, $\mathbf{x}' = (x_1', x_2', x_3')$, transforms q to the form $c_1x_1'^2 + c_2x_2'^2 + c_3x_3'^2$, and also obtain the coefficients of this new form.
c) Is A and thereby q positive definite?

12. If

$$A = \begin{bmatrix} a & b & 0 \\ a & 2 & a \\ 0 & 1 & a \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix},$$

determine for which values of the fixed constants a and b (if any) the system $A\mathbf{x} = \mathbf{c}$ has the following

a) A unique solution.

b) A one-parameter solution.

c) A two-parameter solution.

d) No solution.