Preliminary Examination
Department of Mathematics
Rensselaer Polytechnic Institute
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Notes:

- No books, notes or calculators are allowed.
- Please do any 10 problems. All problems are weighted equally and on a multi-part question, all parts are weighted equally.
- On the front page of the answer book(s), identify the 10 problems you wish graded. Only the 10 problems that you indicate will be considered.
- You have 4 hours to complete the exam.
- Show all work. Justify your answers.
- In some cases, answers to an earlier part of a problem may provide helpful hints on how to solve later parts of a problem.

Preliminary Examination

Problems

1. (a) Find the derivatives of f and g where $f(x) = x^x$ and $g(x) = x^{x^x}$.

   (b) There is a differentiable function $y = f(x)$ that satisfies the equation $x^y + y^2\cos(\pi x) = 0$. Find the value of $\frac{dy}{dx}$ at the point $(x, y) = (1, 1)$.

2. (a) Find the Taylor series of $f(x) = \dfrac{1}{1+x^2}$ about $x_0 = 0$.

   (b) Find the Taylor series of $g(x) = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$ about $x_0 = 0$ and determine its interval of convergence.

   (c) Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}$

3. For each of the functions below, do the following,

   (a) Determine whether the function is differentiable at 0.

   (b) Determine whether the derivative of the function (if it exists) is continuous at 0.

$$f(x) = \begin{cases} x\sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$g(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

4. (a) Suppose f is a function defined on $[a, b]$. State sufficient conditions on f so that there exists a $c \in (a, b)$ so that $f(b) - f(a) = f'(c)(b - a)$. Explain what this result means about the graph of the function f.

   (b) Show that $|\sin(x) - \sin(y)| \leq |x - y|$ for all x and y.

5. A silo is made in the form of a circular cylinder of radius r and height h with a conical cap of height H. If the radius and volume V of the silo are fixed, how should h and H be chosen so as to minimize the surface area? (Hint: the volume and surface area of the cone are $\pi r^2 H/3$ and $\pi r\sqrt{r^2 + H^2}$, respectively.)

6. Let

$$I = \int_\Gamma \left(e^{2y}\cos 3x + \alpha y^2\right) dx + \left(-3 + 2xy + \beta e^{2y}\sin 3x\right) dy$$

where Γ is a piecewise smooth curve from $(0, 1)$ to $(\pi/2, 0)$, and α and β are constants.

   (a) Find α and β such that I is independent of the choice of Γ.

   (b) Evaluate I for the choice of α and β in (a).

7. Suppose $f(x,y,z)$ and $g(x,y,z)$ have continuous partial derivatives of at least second order.

   (a) Find the divergence of $g\nabla f - f\nabla g$.

   (b) Show that

$$\iint_{\partial\Omega} \left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n} \right) dA = \iiint_{\Omega} \left(g\nabla^2 f - f\nabla^2 g \right) dV$$

   where $\partial\Omega$ is a smooth surface which is the boundary of a domain Ω and $\frac{\partial f}{\partial n}$ and $\frac{\partial g}{\partial n}$ are derivatives of f and g, respectively, in the direction of the outward normal vector on $\partial\Omega$.

8. Evaluate

$$\int_0^1 \int_y^1 \frac{e^x - 1}{x}\, dx\, dy, \qquad \int_0^{3/2} \int_{\sqrt{3}x}^{\sqrt{9-x^2}} e^{-x^2} e^{-y^2}\, dy\, dx$$

   It may be helpful to interchange the order of integration or switch to polar coordinates.

9. Let A be a n by n matrix with non-negative entries such that the entries of each row sum to one.

   (a) Give a 5 by 5 example of such a matrix A that has at least 2 non-zeroes in each row.

   (b) Verify directly that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix that you gave in part (a).

   (c) What is the eigenvalue of the eigenvector in (b)?

10. Let

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

   (a) Find a basis for the null space of B.

   (b) Show that the rank of B is 2 and find a basis for the range of B.

   (c) Let $c = \alpha_1 b_1 + \alpha_2 b_2$ where b_1 and b_2 are the vectors in the basis you found in part (b) and α_1 and α_2 are given scalars. Find all solutions to $Bx = c$.

11. Let P be a n by n matrix with non-negative entries.

   (a) Show that $P \cdot P$ is a n by n matrix with non-negative entries.

   (b) Give two different examples of n by n matrices with orthonormal columns and non-negative entries.

12. Let

$$K = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}, \qquad 0 < p, q < 1.$$

   (a) Show that $\begin{bmatrix} qr \\ pr \end{bmatrix}$ is an eigenvector of K for any real value r and show that the associated eigenvalue is one.

   (b) Show that the choice $r = 1/(p+q)$ gives a unique eigenvector whose entries sum to one.