Preliminary Examination

January 8, 2000

Ground Rules: No books, notes, or calculators are permitted. Please select 10 questions out of the following 12 questions to be graded. You have 4 hours to complete the exam.

1. (a) Evaluate the integral

$$\int_0^a \int_{x^2}^{a^2} \sqrt{y} e^{-y^2} \, dy dx.$$

   (b) Consider the integral

$$\iint_D \frac{1}{1+x^2+y^2} \, dA,$$

   where D is the entire xy-plane. Indicate why this is an improper integral. Determine whether it is convergent or divergent. If convergent, evaluate it.

2. (a) Find the Taylor series about $x = 0$ for the function

$$f(x) = \frac{1}{1+x^2}$$

   For what values of x does this series converge?

   (b) Find a Taylor series for $F(x) = \arctan x$. For what values of x does this series converge?

   (c) Noting that $\pi = 6 \arctan 3^{-1/2}$ and $\pi = 4 \arctan 1$, discuss whether either of these equations and the series you generated in part (b) can be used to find an approximation to π. How many terms would be necessary to obtain accuracy to within 0.0001?

3. (a) Let $A := \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$. Show how to calculate A^{20} efficiently.

   (b) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T(3,2) = (1,0)$ and $T(2,1) = (1,1)$ Find a matrix M such that $T(x) = Mx$ for all x.

4. (a) If $y = f(x)$ is defined by $y = x^y$, find y'.

   (b) Let $y = f(x) = |x| + x^3$. What is the largest interval on which an inverse function $x = f^{-1}(y)$ can be defined, if we require that $f^{-1}(0) = -1$? Sketch the graph of the function and its inverse.

5. Let

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \text{ and } b := \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

   (a) For what values of b_1, b_2, and b_3 does the system $Ax = b$ have
      i. a unique solution?
      ii. infinitely many solutions?
      iii. no solutions?

   (b) Find a basis for the nullspace of A.

6. Consider the straight lines

$$\mathcal{L}_1 : \quad x = t,\ y = t,\ z = t,$$

$$\mathcal{L}_2 : \quad x = 1 + s,\ y = 2s,\ z = 2 + 2s.$$

   (a) Show that the lines are skew.

   (b) Express the distance $\mathcal{D}$ between a general point on $\mathcal{L}_1$ and a general point on $\mathcal{L}_2$ as a function of t and s.

   (c) Minimize $\mathcal{D}^2$, and hence locate the points on the two lines that are closest to each other.

7. (a) Define, as mathematical expressions, the terms *gradient*, *linear approximation* and *differential* as they apply to the function $f(x, y, z)$. How are these terms related to one another, and what are their geometric or physical interpretations?

   (b) Compute the above quantities for the function $f(x, y, z) = x^2 \sin y + xye^z$.

   (c) Suppose that a particle moves in space along the path

$$x = 2e^{(t-1)}, y = \pi t/2, z = \ln t.$$

   What value of df/dt does the particle experience at the point $t = 1$, where f is the function defined in part (b) above?

   (d) What is the approximate value of f at $t = 1.001$?

8. (a) Let A be a real $n \times n$ matrix and let x be an n-vector. Show that the following definitions of the matrix norm $\|A\|_2$ are equivalent:
      i. $\|A\|_2 := \max\{\|Ax\|_2 : \|x\|_2 = 1\}$.
      ii. $\|A\|_2 := \max\{\frac{\|Ax\|_2}{\|x\|_2} : x \neq 0\}$.
      iii. $\|A\|_2 :=$ (maximum eigenvalue of $A^TA)^{1/2}$.

(b) The Frobenius norm of a real $n \times n$ matrix is defined as

$$\|A\|_F := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}.$$

Show that $\|A\|_F = \mathrm{tr}(A^T A)$, where $\mathrm{tr}(M)$ denotes the trace of the matrix M.

9. Thompson's Island Pool is a reach of the upper Hudson River that is contaminated with PCBs. The PCBs are trapped in the sediment. Suppose that all the sediment (contaminated and clean) occupies the region $z \geq 0$. Also, assume that borehole sampling determines that there is a "wedge" of contaminated sediment described by $0 \leq x \leq 100(z-3)^2$, $0 \leq y \leq 100$, $0 \leq z \leq 3$ (all of these length measurements are in meters) with uniform concentration 3×10^{-6} gm/cm^3.

   (a) Find the mass of PCBs trapped in this wedge.

   (b) If an erosion event begins at $t = 0$ and erodes 2 cm of sediment per hour, find the rate at which PCBs enter the water column.

10. (a) Find the line tangent to the intersection of $z = x^2 + y^2 - x$ and $x + y + z = 3$ at the point $(1, 1, 1)$.

    (b) For the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - x + x^2 = 0$$

    i. Find all steady state solutions.

    ii. Determine whether the steady state solution $x \equiv 1$ is stable.

11. Let V be the real vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$. Which of the following sets of functions are subspaces of V?

    (a) All f such that $f(x^2) = f(x)^2$.

    (b) All f such that $f(0) = f(1)$.

    (c) All f such that $f(2) = 1 + f(1)$.

    (d) All f which are continuous.

12. The figure below shows a triangular figure, formed in the first octant by the intersection of an oblique plane with the coordinate planes.

    (a) Find an equation for the oblique plane, and an expression for the unit normal to the plane. Make sure that the direction of the normal is consistent with the orientation of the bounding curve C.

(b) Compute the curl of $\underline{F} =< y+z, x-y^3, -x+\sin z >$.

(c) Compute the circulation of $\underline{F}$ along the path C .