Rensselaer Polytechnic Institute
Department of Mathematical Sciences
Preliminary Examination
September 2005

Directions Please solve 10 out of 12 problems, making sure to indicate on the front of the exam book which problems you want to be graded. You have 4 hours to complete the exam. The use of bookes, notes, calculators and other aid is not permitted. To receive a full credit for the problem it should be accompanied by detailed justification of your work.

Good luck,

Preliminary Exam Committee

First Problem Given the fact that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

Find

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)} dx dy.$$

Second problem Calculate

$$I = \int_{0}^{1} \frac{x^3}{\sqrt{4+x^2}} dx.$$

Third problem Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed through the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P.
(a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time.
(b) Is there a limiting concentration at $t \to \infty$, and if yes, what is its value?
(c) If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval before the concentration of pollutants is reduced to the fraction q of its original value.

Fourth problem Find the minimum distance from the surface of the cone, $z = \sqrt{x^2+y^2}$ to the point $(4, 0, 0)$
Solution Minimize

$$(x-4)^2 + y^2 + z^2$$

subject to the constraint

$$z = \sqrt{x^2 + y^2}.$$

Fifth problem A circular dusk of radius 2 with its center at the origin is revolved about the line $y = -6$ to form a solid circular torus. Find the volume of this torus. Be sure to explain how you arrived at your answer.

Sixth problem (a) Find a power series for the hyperbolic sine,

$$\sin(x) = \frac{e^x - e^{-x}}{2},$$

expanded around the point $x_0 = 0$.

(b) Consider the power series

$$\sum_{k=1}^{\infty} a_k x^k = x/2 + x^2/2 + x^3/2^3 + x^4/2^3 + x^5/2^5 + x^6/2^5 + ...$$

(i) explain why the ratio test for the absolute convergence fails to yield an interval of convergence for this series.

(ii) show that the series

$$x + x^2/2 + x^3/2^2 + x^4/2^3 + ...$$

converges

(iii) Use the comparison test to show that the series in part (b) converges for $|x| < 2$.

Seventh problem Given $f(x)$ a differentiable function from R^1 to R^1.

(a) Derive the equation of the secant line that passes through the points $(a, f(a))$ and $(a + h, f(a + h))$.

(b) Derive the equation of the line tangent to the graph of $f(x)$ at $x = a$.

(c) Give an explanation why the derivative of $f(x)$ at $x = a$ is given by

$$f`(a) = \lim h \to 0 \frac{f(a + h) - f(a)}{h}.$$

Eighth problem: You have bought a diamond as an investment. Suppose the value in dollars for the diamond grows over time t according to

$$N(t) = 100 + 10t.$$

If you can get a 5% annual return from a money market account, is there a time in the future when you should sell the diamond and put the dollars in the money market? If so, when?

Ninth problem Let

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & a \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1. For what value of a is A **not** invertible?
2. Let a^* be the value found in part (a). Show that the system $Ax = b$ is inconsistent when $a = a^*$.
3. Let a^* be the value found in part (a). Now let $a = a^* + \epsilon$, where ϵ is a small positive number. Find the solution to the system of equations $Ax = b$. Show that the norm of the solution vector is inversely proportional to ϵ.

Tenth problem The 3×3 matrix A is symmetric. Two eigenvectors of A are

$$u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}.$$

1. Find a matrix Q that orthogonally diagonalizes A.
2. The eigenvalue of u is 1 and the eigenvalue of v is -1. Let

   $$b = \begin{bmatrix} 6 \\ 3 \\ -8 \end{bmatrix}.$$

   Calculate $A^7 b$.
3. Find a symmetric 3×3 matrix A with eigenvector u with eigenvalue 1, with eigenvector v with eigenvalue -1, and with the third eigenvalue equal to 0.

Eleventh problem Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 4 & 3 \end{bmatrix}.$$

1. Find a basis for the range (column space) of A.
2. Find a basis for the nullspace of A.
3. What are the nullity and rank of A?

Twelveth problem Let

$$A = \begin{bmatrix} 6 & 4-2a \\ 4-2a & 3a \end{bmatrix}, \qquad u = s\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad v = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

for parameters a, s, and t.

1. Show that u and v are eigenvectors of A for any choice of a.

2. What is the largest eigenvalue of A when $a = 3$? What is the largest eigenvalue of A when $a = 1$?

3. What value of a minimizes the largest eigenvalue of A?