

Preliminary Examination
September, 2004

Do 10 of the 12 questions, and make sure to indicate on the front of the answer book which problems you want graded. You have 4 hours to complete the exam. Books, notes, calculators, etc are not allowed during the exam.

1. (a) Give a geometric definition of the number $e = 2.71\cdots$ using basic concepts involving the slope of a function.
(b) Use the result in part (a) to show that

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

2. A pipe must be carried horizontally around a corner, from a 10ft wide hallway to a 5ft wide hallway. You may assume the pipe is a line segment and therefore has no thickness. Formulate an optimization model for determining the longest pipe that can be carried around the corner. You do not need to solve the problem but explain how you would proceed to find the length of the longest pipe.

3. Consider the definite integral

$$\int_0^1 \frac{\cos(x) - 1}{x} dx$$

- (a) Find a power series for the integrand that can be used to evaluate the integral.
(b) Write down an infinite series for the exact value of the integral.
(c) If you truncate the series in part (b), how would you determine the error in the resulting approximation for the value of the integral?

4. (a) Find parametric equations for the line that represents the set of all solutions to the system of equations

$$\begin{aligned} x + y - z &= 2 \\ y + z &= 4 \end{aligned}$$

- (b) Consider the subspace of \mathbb{R}^3 given by the span of the two vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}.$$

Find an orthogonal basis for this subspace.

5. A matrix \mathbf{A} is given as

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Find a matrix \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. Verify your claim.

6. Letting

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

set $\mathbf{v}_1 = \mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{v}_2 = \mathbf{e}_1 + 2\mathbf{e}_2$. Also, let B denote the ordered basis for \mathbb{R}^2 given by $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, and let L be a linear transformation on \mathbb{R}^2 that satisfies $L(\mathbf{v}_1) = \mathbf{v}_2$ and $L(\mathbf{v}_2) = \mathbf{v}_1$.

- Find a 2×2 matrix that represents L with respect to the basis B .
- Find an explicit form of the vector $\mathbf{v}_3 = -\mathbf{e}_1 - 11\mathbf{e}_2$, with respect to the basis B .
- Using a matrix multiplication, find an explicit form of the vector $L(\mathbf{v}_3)$ with respect to the basis B .

7. (a) Evaluate the line integral $\int_C 2xydx + x^2dy$ where C is the path in \mathbb{R}^3 given by $\mathbf{x} = \left(2\cos\theta + \cos\frac{\theta}{2}, 2\sin\theta + \cos\frac{\theta}{2}, 1 - \theta \right)$, $0 \leq \theta \leq 4\pi$.

(b) Evaluate the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS$ where $\mathbf{f} = (x, x - y^2, \sin(y) + 2yz)$, S is the surface of the box $1 \leq x \leq 2, 2 \leq y \leq 3, 3 \leq z \leq 4$, and \mathbf{n} is the unit outward normal.

8. (a) The position, velocity and acceleration vectors of a particle moving in \mathbb{R}^3 are, respectively, $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$. Assuming the speed is constant, show that the acceleration vector is orthogonal to the velocity vector for all time.

(b) A point moves around the unit circle in \mathbb{R}^2 in a counter-clockwise direction. Either prove that the acceleration vector always points towards the center of the circle or provide an explicit example where this is not the case.

9. In \mathbb{R}^3 find the distance from the origin to the surface $xyz = 4$. You may assume that there exists a point on the surface closest to the origin and you can leave your answer in the form of a radical.

10. (a) In \mathbb{R}^3 the plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to the ellipse at the point $(x_0, y_0, z_0) = (1, 2, 1)$.

(b) Assuming $xe^{y+z} + z = 1$ can be solved for z as a smooth function of x, y evaluate $\frac{\partial z}{\partial x}$

and $\frac{\partial^2 z}{\partial x^2}$ at $(1, 0, 0)$

11. In \mathbb{R}^3 find the volume of the region where $0 \leq x$, $0 \leq y \leq 4 - x^2$, $0 \leq z \leq 4 - x^2$. This region can be thought of as the intersection, in the first octant, of two cylinders.

12. Letting

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 2 & -4 & 6 & 0 \end{bmatrix}$$

- Find a basis for the null space of \mathbf{A} .
- Determine the nullity and rank of \mathbf{A} .