Preliminary Examination

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
September 6, 2003

Instructions.

- **No calculators, PDAs, computers, books, or cellular phones are allowed.**
- Do not collaborate in any way.
- Do any ten problems.
- All problems are weighed equally: on multi-part questions, all parts are weighed equally.
- On the front page of your answer book(s), identify the ten problems you wish to be graded. Only the ten problems you indicate will be considered.
- In order to receive credit, your answers must be clear, legible, and coherent. Show all your work and justify your answers.
- In some cases, answers to an earlier part of a problem may provide helpful hints on how to solve later parts of a problem.

---

1. Find the Taylor series in the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ for the following functions about the given points $x = c$. Determine the radius of convergence for each series.

   (a)
   $$\frac{4}{2+2x-x^2}, \quad \text{about} \quad x = 1.$$

   (b)
   $$\ln\left(\frac{2+x}{1-x}\right), \quad \text{about} \quad x = 0.$$

2. Consider a population consisting of people who are carriers of a disease (such as typhoid fever) and people who are healthy but susceptible to the disease. Let $x(t)$ and $y(t)$ denote the number of carriers and the number of well (but susceptible) people, respectively, at time t. Suppose these populations satisfy the equations
   $$\frac{dx}{dt} = -\alpha x, \quad \frac{dy}{dt} = -\beta xy,$$
   where α and β are positive constants.

   (a) Find formulas for $x(t)$ and $y(t)$ assuming that $x(0) = x_0$ and $y(0) = y_0$.

   (b) Find a formula, depending on the parameters α, β and x_0, for the fraction of the population of well people that survive the disease in the limit as $t \to \infty$.

3. Let
$$f(x,y) = 4x^2 - xy + y^2 + y.$$

   (a) Find the direction in the xy-plane for which f decreases most rapidly at $x = -2, y = 1$.

   (b) Find the point (or points) in the region $x^2 + y^2/4 \leq 1$ where f attains its absolute maximum value.

4. The circle $x^2 + y^2 = a^2$ is rotated about the line $x = b$, where $b > a$, to form a torus (donut). Find the volume of the solid.

5. A transformation in $\mathbf{R}^2$ consists of stretching all nonzero vectors by a factor of 2 and then rotating the result through 45^o. Find a matrix representation for this transformation.

6. Let
$$A = \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$
   where α and β are real positive constants. For each of the following find all (positive) values. of α and β, if any, so that the statement is true.

   (a) The rank of A is 2.

   (b) A is invertible.

   (c) A is diagonalizable.

7. Transform the Laplacian
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};$$
   into polar coordinates.

8. Evaluate the line integral
$$\int_C \left(y + \frac{1}{1+2x}\right) dx + \left(x + 10y\cos(1+y^2)\right) dy,$$
   where C is the cycloid $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$ from $(0,0)$ to $(2\pi, 0)$.

9. Let
$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 1 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}.$$

   (a) Is there a matrix M such that $A = MB$?

   (b) Is there a matrix C such that $CA = B$?

10. Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) If $B = A^4 - 2kA^2 + k^2 I$, I the identity matrix, determine for what real values of k (if any) the matrix B is positive definite.

(b) If

$$C = \cos(\pi A) = I - \frac{\pi^2 A^2}{2!} + \frac{\pi^4 A^4}{4!} + \cdots,$$

determine the entries of the matrix C.

11. Of the two integrals

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_y^\infty e^{-x^2} dxdy \quad \text{and} \quad \int_0^\infty \int_y^\infty \frac{\sin x}{x} dxdy.$$

one converges and one diverges. Demonstrate which one diverges and evaluate the one that converges.

12. (a) Evaluate the limits

$$\lim_{x \to 0+} x^{1/x} \qquad \lim_{x \to \infty} x^{1/x}.$$

(b) Find the maximum of $f(x) = x^{1/x}$, $x > 0$. Sketch the graph of $f(x)$.