Preliminary Examination

Department of Mathematical Sciences
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Notes:

- No books, notes or calculators are allowed.
- Please do any 10 problems. All problems are weighted equally.
- On the front page of the answer book(s), identify the 10 problems you wish graded. Only the 10 problems that you indicate will be considered.
- You have 4 hours to complete the exam.
- Show all work. Justify your answers.
- In some cases, answers to an earlier part of a problem may provide helpful hints on how to solve later parts of a problem.
- The transpose of a matrix A, for example, is denoted by A^T.
- $\int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi}$

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1. Consider the $n \times n$ matrix
$$H = I - 2\frac{vv^T}{v^Tv},$$
where I is the $n \times n$ identity, v is a (nonzero) n-vector and v^T is its transpose.

   (a) Show that H is an orthogonal matrix.

   (b) Show that $\lambda = -1$ is an eigenvalue of H.

2. (a) Determine whether the following improper integral converges or diverges. Justify your answer.
$$I = \int_0^{\infty} \left(\frac{x}{1 + x^{7/3}}\right) dx$$

   (b) Determine an equation for the line tangent to the curve given by $f(x, y) = x \ln y + x^{2y} = 1$ at the point $(1, 1)$.

3. A tank with a capacity of 50 gal originally contains 20 gal of water with 10 lb of salt in solution. Water containing 0.1 lb of salt per gallon enters the tank at a rate of 3 gal/min, and the well-mixed solution flows out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

4. Let $\mathbf{v} = [1, -2, -4]^T$ and let W be a subspace of R^3 defined by

$$W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ x_1 + x_2 - 3x_3 = 0\}$$

(a) Find an orthonormal basis for W.

(b) Find a vector $\hat{\mathbf{w}} \in W$ such that $(\mathbf{v} - \hat{\mathbf{w}})^T \mathbf{w} = 0$ for all vectors $\mathbf{w} \in W$.

5. Find the absolute minimum and maximum of the function

$$f(x, y) = x^2 + y^2 - x - y + 1$$

for all (x, y) in the disk $x^2 + y^2 \leq 1$.

6. Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x, y, z) will be given by $T(x, y, z) = \exp(-x^2 - 2y^2 - 3z^2)$, where x, y, and z are measured in meters. He is currently at $(1, 1, 1)$.

(a) In what direction should he proceed in order to decrease the temperature most rapidly?

(b) If the ship travels at e^8 meters per second, how fast will the temperature decrease if he proceeds in that direction?

(c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14e^2}$ degrees per second. Describe the set of directions in which he may proceed to bring the temperature down at no more than that rate.

7. Find the Taylor series of the given functions about the designated points. In each case, find the interval of convergence of the series.

(a) $f(x) = \dfrac{1}{x^2 + x + 1}$ about $x_0 = -\frac{1}{2}$.

(b) $g(x) = \ln(2 - x)$ about $x_0 = 0$.

8. The distribution of mass (i.e., the surface density) on the semispherical shell $z = (R^2 - x^2 - y^2)^{1/2}$ is given by

$$\sigma(x, y, z) = (\sigma_0 / R^2)(x^2 + y^2)$$

where σ_0 is constant. Find the total mass of the shell.

9. Let A be a real, nonsingular $n \times n$ matrix.

(a) If A is symmetric, show that A^{-1} is symmetric.

(b) If A is upper triangular, show that A^{-1} is upper triangular.

10. Assume that f and g have continuous partial derivatives of at least the second order. Let $D \subset R^3$ be a bounded domain with the smooth surface S.

(a) Use the divergence theorem to show that

$$\iint_S g \frac{\partial f}{\partial n}\, d\sigma = \iiint_D \left(g\nabla^2 f + \nabla f \cdot \nabla g\right)\, dV$$

(b) Show that if $\nabla^2 f = 0$ in D, then

$$\iint_S \frac{\partial f}{\partial n}\, d\sigma = 0$$

11. Let A be a real $n \times n$ matrix with the property that

$$\langle A\mathbf{x}, \mathbf{x}\rangle > 0, \qquad \text{for any } \mathbf{x} \in R^n,\ \mathbf{x} \neq 0$$

where $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = \sum x_i y_i$ is the standard inner product in R^n.

(a) Is A necessarily symmetric?

(b) Assuming that A is symmetric prove that all eigenvalues of A are real and positive, and that the eigenvectors can be chosen perpendicular to each other.

(c) Assuming that A is symmetric prove that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\langle A\mathbf{x}, \mathbf{x}\rangle}\, dx_1 \cdots dx_n = \pi^{n/2}/\sqrt{\det A}$$

12. In the figure below, the line $x = 1$ is tangent to the unit circle $x^2 + y^2 = 1$ at the point $A(1, 0)$. Given a point Q on the line $x = 1$, select P on the circle so that the length of the line segment AQ is equal to the length of the arc AP. Let R be the point of intersection of the x-axis and the line PQ. Find the length of OR in the limit as P approaches A.