Preliminary Examination

September 2001

NOTES

1. No books, notes or calculators are allowed.
2. Please do any 10 problems. All questions are weighted equally, and in a multi-part question, all parts are weighted equally.
3. On the front page of the answer book, identify the 10 problems you wish to have graded.
4. You have 4 hours to complete the examination.
5. In the following, the adjoint (complex-conjugate transposed) of a matrix A is denoted by A^*.

PROBLEMS

1. (a) Find the Taylor Series for the function

$$f(x) = \frac{1}{(1-x)^2}$$

   about the point $x_0 = 0$. For which values of x does this Taylor series converge?

   (b) By using the result of part (a) or otherwise, show that the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges and find its sum.

2. Suppose α is a real constant and define

$$f(x) = \begin{cases} |x|^\alpha \sin(1/x^2) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) For which values of α is f differentiable at $x = 0$? Explain.

   (b) For which values of α is f twice differentiable at $x = 0$? Explain.

3. Determine whether the following improper integrals converge or diverge. Justify your answers.

   (a) $\int_1^\infty \frac{\cos(x)}{x^2} dx$

   (b) $\int_1^\infty \frac{\sin(x)}{x} dx$

4. The vertical wall of a house is perpendicular to the flat ground. A 10 foot ladder leans against the house, with its top in contact with the wall and its base resting on the ground. Suppose that the base of the ladder, while remaining in contact with the ground, is pulled away from the house at a constant velocity of 2 feet/second.

   (a) How fast is the top of the ladder moving vertically when it is at a distance y, measured in feet, above the ground. Give your answer in terms of y.

   (b) According to your answer to part (a), what is the limit of the vertical velocity of the top of the ladder as $y \to 0$?

   (c) Either defend your answer to part (b), or suggest which of the underlying assumptions need to be abandoned and why. If you can, propose a more realistic mathematical model of the motion of the ladder.

5. A line L lies in a plane M. Let P be a point not on M. Let P_1 be the projection of P on L. Let P_2 be the projection of P on M, and P_3 be the projection of P_2 on L. Must P_1 and P_3 be always the same point? Justify your answer.

6. (a) The velocity of a moving particle is given by $\vec{v}(t) = \sin t\, \vec{\imath} + \cos t\, \vec{\jmath} + \frac{1}{2}\, \vec{k}$. Find the change in the position of the particle, and the total distance travelled by the particle, from $t = 0$ to $t = \pi$.

   (b) Let x and y be the two sides of a triangle and θ the included angle measured in radians. Let A be the area of the triangle.

      i. To changes in which of the three quantities, x, y and θ, is A most sensitive when $\theta = \pi/2$ and $x > y$?

      ii. Let $\theta = \pi/4$. For what values of x and y will it be the case that A is most sensitive to changes in θ?

   Justify your answers.

7. Let the elevation of a hilly region be given as

$$h(x, y) = 5000 - (x^2 + 2y^2 - 2xy)/100$$

as a function of the horizontal coordinates (x, y). Suppose that you have in hand a topographic map of the region, on which the elevation is represented by contour lines. Points on this map have coordinates (x, y).

   (a) Find the gradient of h at the point $P(1, 2)$.

   (b) If one travels from P towards $Q(4, 4)$, what rate of change of elevation does one encounter?

   (c) Find, as a unit vector, the direction at P along which the elevation drops most rapidly. Also find the rate of this drop.

   (d) Starting at P, suppose that we wish to plot a curve on the map, along which the elevation drops at a constant rate r (change in elevation per unit horizontal distance travelled). Assume that the curve can be expressed as $y = f(x)$. Indicate how you would go about finding $f(x)$ analytically. (It is enough to produce one or more (differential) equations for $f(x)$; you need not attempt a solution of these equations.)

8. We wish to find the average distance from the origin to points inside the sphere $x^2 + y^2 + z^2 = 1$.

   (a) Give an expression for the desired quantity in terms of one or more multiple integrals.

   (b) Expressing these integrals in an appropriate spherical coordinate system, evaluate this quantity.

9. For the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

find

   (a) a basis for the <u>null</u> space of A,

   (b) a basis for the <u>range</u> of A, and

   (c) <u>all</u> solutions x to

$$Ax = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

10. Consider the matrix

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and an orthonormal basis of eigenvectors for B.

(b) Find a unitary matrix U such that $U^*BU = D$ where D is diagonal. Also find the determinant of e^B.

11. Given a real, $m \times n$ matrix A and vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}.$$

(a) Prove that if x minimizes

$$\|Ax - b\|^2 \equiv (Ax - b)^*(Ax - b),$$

then

$$A^*Ax = A^*b. \qquad (1)$$

(b) Find the minimum-norm solution to (1) above in terms of the eigenvectors and eigenvalues of A^*A.

12. Let $A = A^*$ be an $n \times n$ self-adjoint matrix, and let λ and the unit vector v be guesses for an eigenvalue and the corresponding eigenvector of A, respectively.

(a) Show that

$$\min|\lambda - \lambda_j| \leq \|Av - \lambda v\|,$$

where the λ_j are the eigenvalues of A.

(b) Find the λ that minimizes $\|Av - \lambda v\|$.