# Preliminary Examination Sample

The purpose of the examination is to assess the qualifications of students in critical areas of undergraduate mathematics. The exam will consist of 12 questions, about one third from linear algebra and about two thirds from basic calculus. Students will have to choose 10 questions to be graded. The duration of the test will be 4 hours. The level of difficulty of questions is very close to the level of the standard GRE mathematics test. The linear algebra and calculus questions from GRE preparation books can serve as good practice questions for the test. The format of the test is different from the GRE test: there is no multiple choice questions.

The following content descriptions are copied from the GRE official publication.

**Calculus**: The usual material of calculus, including trigonometry, coordinate geometry, introductory differential equations, and applications based on calculus.

**Linear algebra**: Matrices, linear transformations, characteristic polynomials, eigenvectors, and other standard material.

Some idea about the scope and depth of the examination can be obtained from the following problems which comprized an actual test.

**Please select 10 questions out of the following 12 questions to be graded. The duration of the test is 4 hours.**

1. Let $V$ be the space of real polynomials of degree 2 on [0,1]. For $f, g \in V$ introduce the scalar product by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

(a) The polynomials $p_0(x) = 1$ and $p_1(x) = x - \frac{1}{2}$ are orthogonal. Find a polynomial $p_2(x)$ such that $\{p_0, p_1, p_2\}$ is an orthogonal basis in $V$.
(b) Let $D$ be the operator of differentiation in $V$, $D : f(x) \to df/dx$. Find the matrix of $D$ relative to the basis found in Part (a).

2. (a)Compute $M^{100}$, where

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(b)Let $T$ be the transformation of the $xy$-plane that reflects each vector through the $x$-axis and then doubles the vector's length. Is $T$ linear? If the answer is "yes", find the matrix of $T$; if the answer is "no", explain why.

3. Let

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 2 & 9 & -1 & 1 & 13 \\ 0 & 5 & 5 & 2 & 1 \\ 1 & 7 & 2 & 1 & 8 \end{bmatrix}$$

(a) Find the rank of $A$. (b) Find a basis for the null space of $A$. (c) What is the dimension of the range space of $A$?

4. For all the following statements, if the statement is true explain why, if the statement is false, demonstrate a counter example.
(a) An $n \times n$ matrix has the same eigenvalues as its transpose.
(b) An $n \times n$ matrix has the same set of eigenvectors as its transpose.
(c) If two $n \times n$ matrices have the same set of eigenvalues then the matrices are similar.

5. Find the Taylor series expansions for
(a)

$$f(x) = \sinh x$$

(b)

$$f(x) = \int_0^x \frac{\sin t}{t}\, dt$$

(c)

$$f(x) = \frac{1}{(1+3x)^2}$$

6. Let $f(x, y)$ be continuously differentiable.
(a) Find the derivative of $f$ at (1,2) in the direction of the appropriate level curve of $g(x, y) = x^2 - y$.
(b) Show that the maximal rate of change of $f$ occurs in the direction of the gradient of $f$.

7. Determine whether the following integrals converge or diverge. Justify your answer.
(a)

$$\int_0^\infty \frac{e^{-t}}{t}\, dt$$

(b)

$$\int_0^\infty \frac{\sin x}{1+x^2}\, dx$$

(c)

$$\int_0^\infty \sin x^2 \, dx$$

8. Let $f_0(x) = f(x) = 4x - x^3, \ f_1(x) = f(f_0(x)), \ \ldots, f_k(x) = f(f_{k-1}(x))$.
(a) Find $f_{10}'(2)$.
(b) Find

$$\int_{-2}^{2} f_{10}(x) \, dx$$

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9. (a) Find the maximum of the quadratic form

$$Q(x, y, z) = 2x^2 + y^2 + z^2 + 4yz$$

under the constraint $x^2 + y^2 + z^2 = 1$.
(b) Let $f(x) = |x| + 3x^2$. Find $f'(-1)$.

10. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min and the mixture is allowed to flow out of the tank at a rate 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

11. (a) Let $T(x, y)$ solve Laplace's equation, $T_{xx} + T_{yy} = 0$. If $u = u(x, y), \ v = v(x, y)$ is a smooth change of variables, such that $u_x = v_y, \ u_y = -v_x$, and $u_x^2 + u_y^2 \neq 0$, show that $T$ solves Laplace's equation in the new variables as well.
(b) For what real values of $x$ does the following series converge?

$$\sum_{n=0}^{\infty} \frac{e^n (x - \pi)^{2n}}{\sqrt{n^2 + 1}}$$

12. (a) Evaluate the integral

$$\iint_\Omega (2x - y) \, dA$$

where $\Omega$ is bounded by the line $y = x$ and the parabola $x = 2 - y^2$.
(b) On the curve $x = \cos t, y = \sin t, z = t$ find those points where the tangent line is "the most parallel to the $x$-axis". In other words, find the points where the magnitude of the angle between the tangent and the $x$-axis is minimal.